THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
SAYT1134 Towards Differential Geometry
Tutorial 7 (August 23)

1 Fundamental Forms

1.1 First fundamental form (Intrinsic Geometry)

Analogue to the arc length component $\|\mathbf{r}'(t)\|^2 = \langle \mathbf{r}'(t), \mathbf{r}'(t) \rangle$ of a curve, we want to study the local change of a surface S under the local parametrization $\mathbf{x}(u, v)$. In Tutorial 5, we learn the differentiation of a multi-variable function is not a "scalar", but a collection of "partial derivatives" to form a matrix. Define the **first fundamental form** of $\mathbf{x}$ at the point (u, v) by

$$I(u, v) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_v \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix}$$

which is a 2×2 symmetric matrix.

1.2 Second fundamental form (Extrinsic Geometry)

Second Fundamental form does not store any information like area and lengths. It depends on the normal vector $\mathbf{n}$, which is outside of the tangent space. If $\mathbf{x}(u, v)$ is a regular parametrized surface and it is C^2, the **second fundamental form** of $\mathbf{x}$ at the point (u, v) is defined by

$$II(u, v) = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \langle \mathbf{x}_{uu}, \mathbf{n} \rangle & \langle \mathbf{x}_{uv}, \mathbf{n} \rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n} \rangle & \langle \mathbf{x}_{vv}, \mathbf{n} \rangle \end{pmatrix}$$

which is a 2×2 symmetric matrix since by mixed derivative theorem, we have $\mathbf{x}_{uv} = \mathbf{x}_{vu}$.
We observe that $\mathbf{x}_u, \mathbf{x}_v \perp \mathbf{n}$, so $\langle \mathbf{x}_u, \mathbf{n} \rangle = \langle \mathbf{x}_v, \mathbf{n} \rangle = 0$ is a constant. Differentiating both sides with respect to u, v respectively, we get

$$\begin{cases} \langle \mathbf{x}_{uu}, \mathbf{n} \rangle & = -\langle \mathbf{x}_u, \mathbf{n}_u \rangle \\ \langle \mathbf{x}_{uv}, \mathbf{n} \rangle & = -\langle \mathbf{x}_u, \mathbf{n}_v \rangle \\ \langle \mathbf{x}_{vu}, \mathbf{n} \rangle & = -\langle \mathbf{x}_v, \mathbf{n}_u \rangle \\ \langle \mathbf{x}_{vv}, \mathbf{n} \rangle & = -\langle \mathbf{x}_v, \mathbf{n}_v \rangle \end{cases}$$

2 Gaussian curvature (Extrinsic Way)

For a regular parametrized surface S under the local parametrization $\mathbf{x}(u, v)$ which is C^2. The **Gaussian curvature** of the surface S at the point (u, v) is given by

$$K(u, v) = \frac{\det(II)}{\det(I)} = \frac{eg - f^2}{EG - F^2}$$

where I is the first fundamental form and II is the second fundamental form of the surface S under the local parametrization $\mathbf{x}$.

To understand the "intuitive meaning" of Gaussian curvature, we may first look at the sign of K. The following are some simple examples:

- $K > 0$: Sphere
- $K = 0$: Flat Plane, Cylinder
- $K < 0$: Hyperboloid

3 Extrinsic Method to Compute Gaussian curvature

To compute the Gaussian curvature $K(u,v)$ of a surface S at the point (u,v) under a local parametrization $\mathbf{x}$, there are some general strategies to compute.

3.1 Method 1

1. We compute all partial derivatives of $\mathbf{x}$ first, i.e. $\mathbf{x}_u$ and $\mathbf{x}_v$.
2. Then, we compute $\mathbf{x}_u \times \mathbf{x}_v$ to get the normal vector and get $\det(I) = \|\mathbf{x}_u \times \mathbf{x}_v\|^2$.
3. Normalizing $\mathbf{x}_u \times \mathbf{x}_v$, define $\mathbf{n} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$ to be the unit normal vector to S.
4. Compute all second order partial derivatives of $\mathbf{x}$, i.e. $\mathbf{x}_{uu}$, $\mathbf{x}_{uv} = \mathbf{x}_{vu}$ and $\mathbf{x}_{vv}$.
5. Calculate $\langle \mathbf{x}_{uu}, \mathbf{n} \rangle$, $\langle \mathbf{x}_{uv}, \mathbf{n} \rangle = \langle \mathbf{x}_{vu}, \mathbf{n} \rangle$ and $\langle \mathbf{x}_{vv}, \mathbf{n} \rangle$ one by one.
6. Finally, plug into the extrinsic definition of Gaussian curvature and you will get

$$K = \frac{\det(II)}{\det(I)} = \frac{\langle \mathbf{x}_{uu}, \mathbf{n} \rangle \langle \mathbf{x}_{vv}, \mathbf{n} \rangle - \langle \mathbf{x}_{uv}, \mathbf{n} \rangle^2}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2}$$

3.2 Method 2

1. We compute all partial derivatives of $\mathbf{x}$ first, i.e. $\mathbf{x}_u$ and $\mathbf{x}_v$.
2. Then, we compute $\mathbf{x}_u \times \mathbf{x}_v$ to get the normal vector and get $\det(I) = \|\mathbf{x}_u \times \mathbf{x}_v\|^2$.
3. Normalizing $\mathbf{x}_u \times \mathbf{x}_v$, define $\mathbf{n} = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$ to be the unit normal vector to S.
4. Compute all partial derivatives of $\mathbf{n}$, i.e. $\mathbf{n}_u$ and $\mathbf{n}_v$.
5. Calculate $\langle \mathbf{x}_u, \mathbf{n}_u \rangle$, $\langle \mathbf{x}_u, \mathbf{n}_v \rangle = \langle \mathbf{x}_v, \mathbf{n}_u \rangle$ and $\langle \mathbf{x}_v, \mathbf{n}_v \rangle$ one by one.
6. Finally, plug into the extrinsic definition of Gaussian curvature and you will get

$$K = \frac{(-1)^2 \det(II)}{\det(I)} = \frac{\langle \mathbf{x}_u, \mathbf{n}_u \rangle \langle \mathbf{x}_v, \mathbf{n}_v \rangle - \langle \mathbf{x}_u, \mathbf{n}_v \rangle^2}{\|\mathbf{x}_u \times \mathbf{x}_v\|^2}$$

The above two methods are equivalent.

4 Examples

4.1 Sphere

A sphere of radius r and centered at the origin can be parametrized by

$$\mathbf{x}(\phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi),\ 0 < \phi < \pi,\ 0 < \theta < 2\pi$$

Compute

(a) The first fundamental form of the sphere under the local parametrization $\mathbf{x}$.

(b) The second fundamental form of the sphere under the local parametrization $\mathbf{x}$.

(c) The Gaussian curvature of the sphere under the local parametrization $\mathbf{x}$.

Solution.

(a) First, we have

$$\begin{cases} \mathbf{x}_\phi = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi) \\ \mathbf{x}_\theta = (-r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0) \end{cases}$$

and then

$$\begin{aligned} \langle \mathbf{x}_\phi, \mathbf{x}_\phi \rangle &= (r\cos\phi\cos\theta)^2 + (r\cos\phi\sin\theta)^2 + (-r\sin\phi)^2 \\ &= r^2\cos^2\phi(\cos^2\theta + \sin^2\theta) + r^2\sin^2\phi \\ &= r^2(\cos^2\phi + \sin^2\phi) \\ &= r^2 \\ \langle \mathbf{x}_\phi, \mathbf{x}_\theta \rangle &= (r\cos\phi\cos\theta)(-r\sin\phi\sin\theta) + (r\cos\phi\sin\theta)(r\sin\phi\cos\theta) + (-r\sin\phi)(0) \\ &= -r^2\sin\phi\cos\phi\sin\theta\cos\theta + r^2\sin\phi\cos\phi\sin\theta\cos\theta + 0 \\ &= 0 \\ &= \langle \mathbf{x}_\theta, \mathbf{x}_\phi \rangle \\ \langle \mathbf{x}_\theta, \mathbf{x}_\theta \rangle &= (-r\sin\phi\sin\theta)^2 + (r\sin\phi\cos\theta)^2 + 0^2 \\ &= r^2\sin^2\phi(\sin^2\theta + \cos^2\theta) \\ &= r^2\sin^2\phi \end{aligned}$$

Therefore, the first fundamental form required is

$$I = \begin{pmatrix} r^2 & 0 \\ 0 & r^2\sin^2\phi \end{pmatrix}$$

(b) From (a), we have

$$\begin{aligned} \mathbf{x}_\phi \times \mathbf{x}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos\phi\cos\theta & r\cos\phi\sin\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta & 0 \end{vmatrix} \\ &= \begin{vmatrix} r\cos\phi\sin\theta & -r\sin\phi \\ r\sin\phi\cos\theta & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} r\cos\phi\cos\theta & -r\sin\phi \\ -r\sin\phi\sin\theta & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} r\cos\phi\cos\theta & r\cos\phi\sin\theta \\ -r\sin\phi\sin\theta & r\sin\phi\cos\theta \end{vmatrix}\mathbf{k} \\ &= \left(r^2\sin^2\phi\cos\theta, r^2\sin^2\phi\sin\theta, r^2\sin\phi\cos\theta\right) \\ &= r^2\sin\phi\cdot(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\theta) \end{aligned}$$

Therefore, the unit normal vector is

$$\mathbf{n} = \frac{\mathbf{x}_\phi \times \mathbf{x}_\theta}{\|\mathbf{x}_\phi \times \mathbf{x}_\theta\|} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

and then

$$\begin{cases} \mathbf{n}_\phi = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) \\ \mathbf{n}_\theta = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) \end{cases}$$

$$\begin{aligned} \langle \mathbf{x}_\phi, \mathbf{n}_\phi \rangle &= (r\cos\phi\cos\theta)(\cos\phi\cos\theta) + (r\cos\phi\sin\theta)(\cos\phi\sin\theta) + (-r\sin\phi)(-\sin\phi) \\ &= r\cos^2\phi(\cos^2\theta + \sin^2\theta) + r\sin^2\phi \\ &= r(\cos^2\phi + \sin^2\phi) \\ &= r \end{aligned}$$

and similarly

$$\begin{aligned}
\langle \mathbf{x}_\phi, \mathbf{n}_\theta \rangle &= (r\cos\phi\cos\theta)(-\sin\phi\sin\theta) + (r\cos\phi\sin\theta)(\sin\phi\cos\theta) + (-r\sin\phi)(0) \\
&= -r\sin\phi\cos\phi\sin\theta\cos\theta + r\sin\phi\cos\phi\sin\theta\cos\theta + 0 \\
&= 0 \\
&= \langle \mathbf{x}_\theta, \mathbf{n}_\phi \rangle \\
\langle \mathbf{x}_\theta, \mathbf{n}_\theta \rangle &= (-r\sin\phi\sin\theta)(-\sin\phi\sin\theta) + (r\sin\phi\cos\theta)(\sin\phi\cos\theta) + 0(0) \\
&= r\sin^2\phi(\sin^2\theta + \cos^2\theta) \\
&= r\sin^2\phi
\end{aligned}$$

So, the second fundamental form required is

$$II = -\begin{pmatrix} \langle \mathbf{x}_\phi, \mathbf{n}_\phi \rangle & \langle \mathbf{x}_\phi, \mathbf{n}_\theta \rangle \\ \langle \mathbf{x}_\theta, \mathbf{n}_\phi \rangle & \langle \mathbf{x}_\theta, \mathbf{n}_\theta \rangle \end{pmatrix} = \begin{pmatrix} -r & 0 \\ 0 & -r\sin^2\phi \end{pmatrix}$$

(c) By parts (a) and (b), the Gaussian curvature of the sphere is

$$K = \frac{\det(II)}{\det(I)} = \frac{(-r)(-r\sin^2\phi)}{(r^2)(r^2\sin^2\phi)} = \frac{1}{r^2}.$$

Note. From the above, we can see the Gaussian curvature of sphere is everywhere **positive**.

4.2 Flat Plane

A flat plane can be parametrized by

$$\mathbf{x}(u, v) = (u, v, c), u, v \in \mathbb{R}$$

and c is a real constant. Show that the Gaussian curvature of the flat plane is everywhere zero.
Solution.
We first compute the components of its first fundamental form as follows:

$$\begin{cases} \mathbf{x}_u &= (1,\ 0,\ 0) \\ \mathbf{x}_v &= (0,\ 1,\ 0) \end{cases}$$

Then, we have

$$\mathbf{x}_u \times \mathbf{x}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,\ 0,\ 1)$$

and $\mathbf{n} = \mathbf{x}_u \times \mathbf{x}_v = (0,\ 0,\ 1)$.
Since $\mathbf{x}_{uu} = \mathbf{x}_{uv} = \mathbf{x}_{vu} = \mathbf{x}_{vv} = \mathbf{0}$, so the second fundamental form of the flat plane is

$$II = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus, the Gaussian curvature of a flat plane is

$$K = \frac{0}{0^2 + 0^2 + 1^2} = 0$$

Note 1. From the above, we can see the Gaussian curvature of flat plane is everywhere **zero**.
Note 2. By folding a flat plane towards the z-axis, it becomes a cylinder and the Gaussian curvature of a cylinder is also **zero**.

4.3 One-Sheeted Hyperboloid

A one-sheeted hyperboloid has Cartesian equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2}-\dfrac{z^2}{c^2}=1$ can be parametrized by

$$\mathbf{x}(u,v)=(a\cosh v\cos u, a\cosh v\sin u, c\sinh v),\, u,v\in\mathbb{R}$$

and a,c are positive real constants.
Show that the Gaussian curvature of the hyperboloid is everywhere negative.
Solution.

- Step 1: We first compute $\mathbf{x}_u$, $\mathbf{x}_v$.

$$\begin{cases}\mathbf{x}_u=(-a\cosh v\sin u, a\cosh v\cos u, 0)\\ \mathbf{x}_v=(a\sinh v\cos u, a\sinh v\sin u, c\cosh v)\end{cases}$$

- Step 2: Find $\mathbf{x}_u\times\mathbf{x}_v$ and $\|\mathbf{x}_u\times\mathbf{x}_v\|^2$.

$$\begin{aligned}\mathbf{x}_u\times\mathbf{x}_v&=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-a\cosh v\sin u&a\cosh v\cos u&0\\a\sinh v\cos u&a\sinh v\sin u&c\cosh v\end{vmatrix}\\
&=\left(\begin{vmatrix}a\cosh v\cos u&0\\a\sinh v\sin u&c\cosh v\end{vmatrix},-\begin{vmatrix}-a\cosh v\sin u&0\\a\sinh v\cos u&c\cosh v\end{vmatrix},\begin{vmatrix}-a\cosh v\sin u&a\cosh v\cos u\\a\sinh v\cos u&a\sinh v\sin u\end{vmatrix}\right)\\
&=\big(ac\cosh^2 v\cos u, ac\cosh^2 v\sin u, -a^2\sinh v\cosh v(\sin^2 u+\cos^2 u)\big)\\
&=\big(ac\cosh^2 v\cos u, ac\cosh^2 v\sin u, -a^2\sinh v\cosh v\big)\\
\|\mathbf{x}_u\times\mathbf{x}_v\|^2&=(ac\cosh^2 v\cos u)^2+(ac\cosh^2 v\sin u)^2+(-a^2\sinh v\cosh v)^2\\
&=a^2c^2\cosh^4 v(\cos^2 u+\sin^2 u)+a^4\sinh^2 v\cosh^2 v\\
&=a^2\cosh^2 v(c^2\cosh^2 v+a^2\sinh^2 v)\end{aligned}$$

- Step 3: Find the unit normal vector of the Hyperboloid.

$$\mathbf{n}=\frac{\mathbf{x}_u\times\mathbf{x}_v}{\|\mathbf{x}_u\times\mathbf{x}_v\|}=\frac{(c\cosh v\cos u, c\cosh v\sin u, -a\sinh v)}{\sqrt{c^2\cosh^2 v+a^2\sinh^2 v}}$$

- Step 4: Compute $\mathbf{x}_{uu}$, $\mathbf{x}_{uv}$ and $\mathbf{x}_{vv}$. (Of course, we don't want to compute $\mathbf{n}_u,\mathbf{n}_v$)

$$\begin{cases}\mathbf{x}_{uu}=(-a\cosh v\cos u, -a\cosh v\sin u, 0)\\ \mathbf{x}_{uv}=(-a\sinh v\sin u, a\sinh v\cos u, 0)\\ \mathbf{x}_{vv}=(a\cosh v\cos u, a\cosh v\sin u, c\sinh v)\end{cases}$$

- Step 5: Calculate $\langle\mathbf{x}_{uu},\mathbf{n}\rangle$, $\langle\mathbf{x}_{uv},\mathbf{n}\rangle=\langle\mathbf{x}_{vu},\mathbf{n}\rangle$ and $\langle\mathbf{x}_{vv},\mathbf{n}\rangle$ one by one.

$$\begin{aligned}\langle\mathbf{x}_{uu},\mathbf{n}\rangle&=\frac{-ac\cosh^2 v}{\sqrt{c^2\cosh^2 v+a^2\sinh^2 v}}\\
\langle\mathbf{x}_{uv},\mathbf{n}\rangle&=\frac{1}{\sqrt{c^2\cosh^2 v+a^2\sinh^2 v}}(0)\\
&=0\\
&=\langle\mathbf{x}_{vu},\mathbf{n}\rangle\\
\langle\mathbf{x}_{vv},\mathbf{n}\rangle&=\frac{1}{\sqrt{c^2\cosh^2 v+a^2\sinh^2 v}}(ac\cosh^2 v(\cos^2 u+\sin^2 u)-ac\sinh^2 v)\\
&=\frac{ac}{\sqrt{c^2\cosh^2 v+a^2\sinh^2 v}}\qquad(\because \cosh^2 v-\sinh^2 v=1)\end{aligned}$$

- Step 6: Plug into the formula of Gaussian curvature.
  Thus, the Gaussian curvature of the hyperboloid under the parametrization $\mathbf{x}$ is

$$\begin{aligned} K &= \frac{-ac(ac)\cosh^2 v/(c^2\cosh^2 v + a^2\sinh^2 v) - 0}{a^2\cosh^2 v(c^2\cosh^2 v + a^2\sinh^2 v)} \\ &= -\frac{c^2}{(c^2\cosh^2 v + a^2\sinh^2 v)^2} \\ &< 0 \end{aligned}$$

  and the proof is completed.

Note. From the above, we can see the Gaussian curvature of One-sheeted hyperboloid is everywhere **negative**.

5 More on Gaussian curvature

5.1 Curvature of graphs of functions

1. Let $f(x,y), (x,y) \in D \subset \mathbb{R}^2$, be a function with continuous second derivatives. The Gaussian curvature of the graph of $z = f(x,y)$ in rectangular coordinates is

$$K(x,y) = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}$$

2. Let $f(r,\theta), (r,\theta) \in D \subset \mathbb{R}^+ \times (0, 2\pi)$be a function with continuous second derivatives. The Gaussian curvature of the graph of $z = f(r,\theta)$ in cylindrical coordinates is

$$K(r,\theta) = \frac{r^2 f_{rr}(rf_r + f_{\theta\theta}) - (rf_{r\theta} - f_\theta)^2}{(r^2 + r^2 f_r^2 + f_\theta^2)^2}$$

5.2 Gaussian curvature of surfaces of revolution

1. **By graph of function.** Let $f(z), z \in (a,b)$ be a function with continuous second derivative. The Gaussian curvature of the surface obtained by rotating the graph of $x = f(z)$ on the xz-plane about the z-axis is

$$K(z) = -\frac{f''}{f(1 + f'^2)^2}$$

2. **By parametrized curve.** Let $(\varphi(u), \psi(u))$, $u \in (a,b)$ be a regular parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(u), \psi(u))$ on the xz-plane about the z-axis is

$$K(u) = \frac{\psi'(\varphi'\psi'' - \varphi''\psi')}{\varphi(\varphi'^2 + \psi'^2)}$$

3. **By arc length parametrized curve.** Let $(\varphi(s), \psi(s))$, $s \in (a,b)$ be a regular parametrized curve. The Gaussian curvature of the surface obtained by rotating the curve $(x,z) = (\varphi(s), \psi(s))$ on the xz-plane about the z-axis is

$$K(s) = -\frac{\varphi''}{\varphi}$$

The proofs of the above formulas on computing Gaussian curvature are easy, please refer to Lecture Notes Page 106 to Page 109.

6 Orthogonal Parametrization

Let $S = \mathbf{x}(u, v)$ be a regular parametrized surface. The parametrization is said to be **orthogonal parametrization** if $F = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$, i.e. the first fundamental form of $\mathbf{x}(u, v)$ is of the form

$$I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$$

6.1 Example - Helicoid

Verify that the helicoid is parametrized by

$$\mathbf{x}(u, \theta) = (u\cos\theta, u\sin\theta, \theta),\ u, \theta \in \mathbb{R}$$

is an orthogonal parametrization.
Solution. We compute the first derivatives of $\mathbf{x}$ as:

$$\begin{cases} \mathbf{x}_u = (\cos\theta, \sin\theta, 0) \\ \mathbf{x}_\theta = (-u\sin\theta, u\cos\theta, 1) \end{cases}$$

and therefore

$$\begin{aligned} \langle \mathbf{x}_u, \mathbf{x}_\theta \rangle &= (\cos\theta)(-u\sin\theta) + (\sin\theta)(u\cos\theta) + 0(1) \\ &= 0 \end{aligned}$$

Thus, $\mathbf{x}$ is an orthogonal parametrization.

6.2 Intrinsic Way to Compute Gaussian curvature

Let $\mathbf{x}(u, v)$ be a regular parametrized surface. Suppose $F = 0$, i.e. the first fundamental form of $\mathbf{x}(u, v)$ is

$$I = \begin{pmatrix} E & 0 \\ 0 & G \end{pmatrix}$$

Then, the Gaussian curvature of $\mathbf{x}(u, v)$ is

$$K(u, v) = -\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]$$

Note 1. From the above, we can see the Gaussian curvature is only depending on the first fundamental form, so it is an intrinsic information.
Note 2. We will discuss it in detail in the section - Theorema Egregium.

6.2.1 Example - Helicoid

Show that the Gaussian curvature of the **helicoid** parametrized by

$$\mathbf{x}(u, \theta) = (u\cos\theta, u\sin\theta, \theta),\ u, \theta \in \mathbb{R}$$

is

$$K = -\frac{1}{(1 + u^2)^2}.$$

Solution. From the above, the first fundamental form is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & u^2+1 \end{pmatrix}$$

Then, we have

$$\begin{cases} E_\theta &= 0 \\ G_u &= \frac{\partial}{\partial u}(1+u^2) = 2u \end{cases}$$

Plug into the formula of Gaussian curvature, we have

$$\begin{aligned} K &= -\frac{1}{2\sqrt{u^2+1}}\left[\left(\frac{0}{\sqrt{1+u^2}}\right)_\theta + \left(\frac{2u}{1+u^2}\right)_u\right] \\ &= -\frac{1}{\sqrt{u^2+1}}\left(\frac{\sqrt{1+u^2} - u\left(\frac{u}{\sqrt{1+u^2}}\right)}{1+u^2}\right) \\ &= -\frac{1}{(1+u^2)^2} \end{aligned}$$

6.2.2 Lecture Notes Exercise 3 Q12

Find the Gaussian curvature of the parametrized surface $\mathbf{x}(u, v)$ with the following first fundamental forms.

(a) $I = \begin{pmatrix} \frac{1}{u^2} & 0 \\ 0 & \frac{1}{v^2} \end{pmatrix}$

(b) $I = \begin{pmatrix} \frac{1}{u^2+v^2+1} & 0 \\ 0 & \frac{1}{u^2+v^2+1} \end{pmatrix}$

(c) $I = \begin{pmatrix} 1 & 0 \\ 0 & \cosh^2 u \end{pmatrix}$

6.2.3 Lecture Notes Exercise 3 Q13

Suppose the first fundamental form of a parametrized surface $\mathbf{x}(u, v)$ is given by

$$I = \begin{pmatrix} f^2 & 0 \\ 0 & f^2 \end{pmatrix}$$

where $f = f(u, v) > 0$ is a twice-differentiable function. Show that the Gaussian curvature of the surface is

$$K = -\frac{1}{f^2}\Delta \ln f$$

where $\Delta := \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$ is called the **Laplacian**.

7 Gauss map

7.1 Definition

Let S be a regular surface in $\mathbb{R}^3$ with regular parametrization $\mathbf{x}(u,v)$. For each point $p = \mathbf{x}(u,v)$, we associate the unit normal vector $\mathbf{n}(p)$ to p. This defines a map $\mathbf{n} : S \to \mathbb{S}^2$ from the surface S to the unit sphere $\mathbb{S}^2 = \{(x,y,z) : x^2+y^2+z^2 = 1\}$ and it is called the **Gauss map** of S at the point p.

7.2 Example

7.2.1 Example 1

Show that the Gauss map of a regular parametrized surface S to $\mathbb{S}^2$ is well-defined.

Proof. Let $\mathbf{x}(u,v)$ be a local parametrization of S.
Since S is a regular surface, so by the definition, we have $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$ for any points (u,v).
We can define a map $\mathbf{n} : S \to \mathbb{S}^2$ by $\mathbf{n}(\mathbf{x}(u,v)) = \dfrac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$ which satisfies

$$\|\mathbf{n}(\mathbf{x}(u,v))\| = 1$$

and thus $\mathbf{n}(\mathbf{x}(u,v)) \in \mathbb{S}^2$.

7.2.2 Example 2

A **Menn's surface** $\Phi : (0,2\pi) \times (-1,1) \to \mathbb{R}^3$ is parametrized by

$$\Phi(u,v) = (u, v, au^4 + u^2v - v^2),\ (u,v) \in \mathbb{R}$$

where a is a constant. Find the Gauss map of the Menn's surface at the point $p = \Phi(u,v)$.

Solution. First, we compute

$$\begin{cases} \Phi_u &= (1, 0, 4au^3 + 2uv) \\ \Phi_v &= (0, 1, u^2 - 2v) \end{cases}$$

Next, we find the direction of the normal vector to the Menn's surface as:

$$\begin{aligned} \Phi_u \times \Phi_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 4au^3 + 2uv \\ 0 & 1 & u^2 - 2v \end{vmatrix} \\ &= \left(-4au^3 + 2uv, 2v - u^2, 1\right) \end{aligned}$$

and

$$\|\Phi_u \times \Phi_v\| = \sqrt{(-4au^3 + 2uv)^2 + (2v - u^2)^2 + 1}$$

Thus, the Gauss map of the Menn's surface at the point $p = \Phi(u,v)$ is

$$\mathbf{n}(p) = \frac{1}{\sqrt{(-4au^3 + 2uv)^2 + (2v - u^2)^2 + 1}} \left(-4au^3 + 2uv, 2v - u^2, 1\right)$$

Note that $\mathbf{n}_u, \mathbf{n}_v$ are lying on the tangent space spanned by $\mathbf{x}_u, \mathbf{x}_v$ as well as both of them are orthogonal to $\mathbf{n}$, so the cross product $\mathbf{n}_u \times \mathbf{n}_v$ is normal to the surface S as well and there is a scalar multiple of $\mathbf{x}_u \times \mathbf{x}_v$. This multiple is exactly the Gaussian curvature.

8 Gauss Map and Gaussian Curvature

Let $\mathbf{x}(u, v)$ be a regular parametrized surface and $\mathbf{n}(u, v)$ be the unit normal vector at $\mathbf{x}(u, v)$. Then

$$\mathbf{n}_u \times \mathbf{n}_v = K\mathbf{x}_u \times \mathbf{x}_v$$

where K is the Gaussian curvature of the surface.

8.1 Examples (Lecture Notes Exercise 3 Q20)

Let $\mathbf{x}(u, v)$ be a regular parametrized surface and $\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$ be the unit normal vector. Since $\mathbf{n}_u, \mathbf{n}_v$ are coplanar with $\mathbf{x}_u, \mathbf{x}_v$, so from what we learned in Tutorial 2, we can express $\mathbf{n}_u, \mathbf{n}_v$ as a linear combination of $\mathbf{x}_u, \mathbf{x}_v$, we write

$$\begin{cases} \mathbf{n}_u & = a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v \\ \mathbf{n}_v & = a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v \end{cases}$$

(a) Prove that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = -(II)(I)^{-1}$$

Note. This is the matrix representation of the differential of the Gauss map with respect to the basis $\mathbf{x}_u, \mathbf{x}_v$.

(b) Show that

$$\mathbf{n}_u \times \mathbf{n}_v = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mathbf{x}_u \times \mathbf{x}_v = \frac{eg - f^2}{EG - F^2}\mathbf{x}_u \times \mathbf{x}_v$$

Solution.

(a) First, we compute $\langle \mathbf{n}_u, \mathbf{x}_u \rangle, \langle \mathbf{n}_u, \mathbf{x}_v \rangle$ and $\langle \mathbf{n}_v, \mathbf{x}_v \rangle$ one by one.

$$\begin{cases} \langle \mathbf{n}_u, \mathbf{x}_u \rangle & = a_{11} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + a_{12} \langle \mathbf{x}_v, \mathbf{x}_u \rangle \\ \langle \mathbf{n}_u, \mathbf{x}_v \rangle & = a_{11} \langle \mathbf{x}_u, \mathbf{x}_v \rangle + a_{12} \langle \mathbf{x}_v, \mathbf{x}_v \rangle \\ \langle \mathbf{n}_v, \mathbf{x}_u \rangle & = a_{21} \langle \mathbf{x}_u, \mathbf{x}_u \rangle + a_{22} \langle \mathbf{x}_v, \mathbf{x}_u \rangle \\ \langle \mathbf{n}_v, \mathbf{x}_v \rangle & = a_{21} \langle \mathbf{x}_u, \mathbf{x}_v \rangle + a_{22} \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{cases}$$

and then we rewrite the above system as

$$\begin{cases} \begin{pmatrix} \langle \mathbf{n}_u, \mathbf{x}_u \rangle \\ \langle \mathbf{n}_v, \mathbf{x}_u \rangle \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle \end{pmatrix} \\ \begin{pmatrix} \langle \mathbf{n}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{n}_v, \mathbf{x}_v \rangle \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} \end{cases}$$

Then, we put these column vectors together gives

$$\underbrace{\left(\begin{array}{c|c} \langle \mathbf{n}_u, \mathbf{x}_u \rangle & \langle \mathbf{n}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{n}_v, \mathbf{x}_u \rangle & \langle \mathbf{n}_v, \mathbf{x}_v \rangle \end{array} \right)}_{-II} = \left(\begin{array}{c|c} \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle \end{pmatrix} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix} \end{array} \right)$$

and right-hand side is equivalent to the matrix multiplication

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \underbrace{\begin{pmatrix} \langle \mathbf{x}_u, \mathbf{x}_u \rangle & \langle \mathbf{x}_u, \mathbf{x}_v \rangle \\ \langle \mathbf{x}_v, \mathbf{x}_u \rangle & \langle \mathbf{x}_v, \mathbf{x}_v \rangle \end{pmatrix}}_{I}$$

Thus, we have

$$\begin{aligned} -II &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} I \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= -(II)(I)^{-1} \end{aligned}$$

(b) By direct computation, we have

$$\begin{aligned} \mathbf{n}_u \times \mathbf{n}_v &= (a_{11}\mathbf{x}_u + a_{12}\mathbf{x}_v) \times (a_{21}\mathbf{x}_u + a_{22}\mathbf{x}_v) \\ &= a_{11}a_{21} \underbrace{\mathbf{x}_u \times \mathbf{x}_u}_{\mathbf{0}} + a_{11}a_{22}\mathbf{x}_u \times \mathbf{x}_v + a_{12}a_{21}\mathbf{x}_v \times \mathbf{x}_u + a_{12}a_{22} \underbrace{\mathbf{x}_v \times \mathbf{x}_v}_{\mathbf{0}} \\ &= a_{11}a_{22}\mathbf{x}_u \times \mathbf{x}_v - a_{12}a_{21}\mathbf{x}_u \times \mathbf{x}_v \\ &= \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \mathbf{x}_u \times \mathbf{x}_v \end{aligned}$$

and from part (a), we have

$$\begin{aligned} \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} &= \det \left[(-1)(II)(I)^{-1} \right] \\ &= (-1)^2 \cdot \frac{\det(II)}{\det(I)} \quad (\because I, II \text{ are square matrices}) \\ &= \frac{eg - f^2}{EG - F^2} \end{aligned}$$

Thus, the proof is done.

Note 1. The above matrix representation of the "**negative differential of Gauss map**" has an alternative name, called the "**Shape operator**".

Note 2. In Tutorial 8 and afterwards, we will use **shape operator** to compute other curvatures and study both extrinsic and intrinsic geometry of a surface, for example Mean curvature, Principal curvatues, Normal curvature, etc.